

Finite Element Free Vibration Analysis of Clamped Functionally Graded Plate

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Abstract

The main objective of this research is to present analytical solutions for free vibration analysis of moderately thick plates composed of functionally graded materials. Finite element analysis using first-order shear deformation plate theory (FSDT) has been used in the research. Nine noded-isoparametric lagrangian plate bending elements are used in the analysis. Young's modulus and density per unit volume are assumed to vary continuously through the plate thickness according to power-law distribution. The results have been validated with the existing literature available from other analytical and numerical techniques. The effect of different plate parameters such as aspect ratios, thickness to length ratios, and gradient indices on the natural frequencies of FG rectangular plates is presented. Only clamped boundary conditions have been investigated. It is found that the frequency parameter depends considerably on the said parameters.

Keywords: Functionally graded, Free vibration, Finite element, First order shear deformation

1. Introduction

Functionally graded materials (FGMs) are a type of novel composite material with varying material properties through their thickness. FGMs were adopted first in Japan in 1984 in the form of thermal barrier in a space plane project [1]. Thereafter this concept became most popular in Europe and Germany. Different types of materials are considered as FGM materials i.e. steel, aluminium, polypropylene etc. Due to gradation of material properties and thermal resistance behaviour, there are a wide variety of applications of these materials in different sectors such as in defence sector, automobile and aerospace engineering, naval structure and in medical field. Structures made up of laminated composites have high interlaminar stresses due to sudden change in material property across the interface which results in delamination. In case of FGMs, due to smooth and continuous varying material properties from one face to other, this problem can be overcome and hence FGMs are usually superior to the conventional composite materials in mechanical behaviour. FGMs are typically made of a mixture of ceramic and metal and can resist high thermal load while maintaining toughness too.

In 2000, Reddy [2] developed a theoretical formulation and finite element model based on third order shear deformation theory (TSDT) to analyse FGM plate structure. Velet *et al.* [3] obtained the exact solution for a simply supported thin and thick FGM rectangular plate by employing FSDT and TSDT. Batra *et al.* [4] studied the free vibration analysis of the

rectangular anisotropic FGM plate by using FSDT with finite element method. Hosseini-Hashemi *et al.* [5] obtained an exact closed-form frequency equation for free vibration analysis of circular and annular moderately thick FG plates based on the Mindlin's theory. Efrain and Eisenberger [6] obtained the equations of motion for annular FGM plate using FSDT and solved the vibration frequencies and modes for various boundary conditions. Nguyen *et al.* [7] described an FSDT model in which the transverse shear factors are obtained using energy equivalence methods. Talha and Singh [8] studied static response and free vibration analysis of FGM plates using higher order shear deformation theory. Hoisseini and Hashemi [9] proposed a new exact analytical approach for free vibration analysis of a rectangular moderately thick FGM plate having two opposite edges simply supported using FSDT. Thai *et al.* [10] presented a simple FSDT for bending and free vibration analysis for FGM plates and concluded the same accuracy with conventional FSDT with least numbers of unknowns. Li *et al.* [11] analysed the free vibration of functionally graded porous spherical shell using FSDT. It was shown that the proposed method has advantages of fast convergence, high calculation efficiency, high solution accuracy and simple boundary simulation.

In this paper free vibration analysis of moderately thick functionally graded plates with clamped boundary conditions has been shown. Due to high efficiency and simplicity, first-order shear deformation plate theory has been used in the

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research. Nine noded isoparametric lagrangian plate bending elements are used in the analysis. Young's modulus and density per unit volume are assumed to vary continuously through the plate thickness according to power-law distribution. The effect of different plate parameters such as aspect ratios, thickness to length ratios, different materials and gradient indices on the natural frequencies of FG rectangular plates is presented.

2. Theoretical Formulation

2.1. Geometric configuration and material properties

The FGM plate with dual material properties continuously varied throughout the thickness are considered. The composition is varied from top surface ($z = h/2$) with ceramic to bottom surface ($z = -h/2$) with a metal as shown in Fig. 1. Linear elastic material behaviour with small displacements and strains are assumed in the analysis.

The profile for volume fraction is taken from power law distribution expressed as below:

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^N, \quad V_m + V_c = 1 \quad (1)$$

where N is the power law index. The material properties i.e., modulus of elasticity, shear modulus, density and poisson's ratio are function of z and given as

$$P(z) = P_c V_c + P_m V_m \quad (2)$$

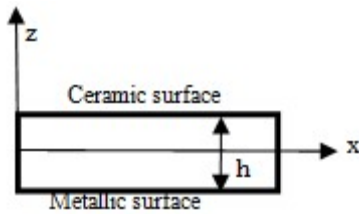


Fig. 1: Geometry of FGM plate

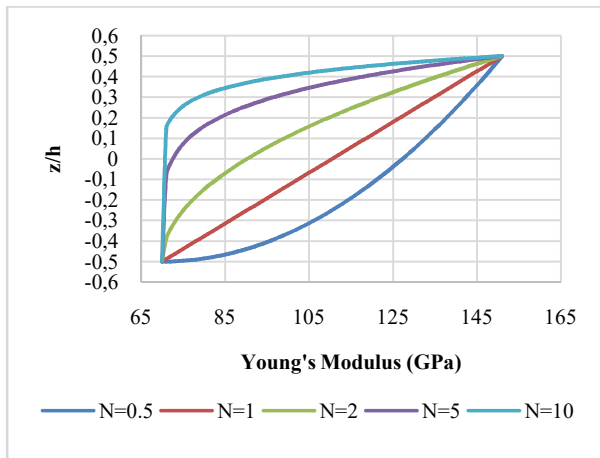


Fig.2. Variation of Young's modulus (Al/ZrO₂) with non-dimensional thickness for different values of the N

Table-1. Material properties of the used FG plate

Material	Properties		
	E(GPa)	ν	ρ (kg/m ³)
Aluminium(Al)	70	0.3	2707
Alumina(Al ₂ O ₃)	380	0.3	3800
Zirconia(ZrO ₂)	151	0.3	3000

Where z is the thickness direction, P_c, V_c and P_m, V_m are the material properties and volume fractions of ceramic and metallic respectively. Fig. 2 indicates the change in variation in modulus of elasticity with change in power law index N . $N=0$ indicates pure ceramic and $N=\infty$ indicates pure metallic.

2.2. Displacement fields and strains

In the present finite element analysis first order transverse shear deformation theory with constant transverse shear strain through the plate thickness has been used. Effect of rotary inertia is also taken in the element mass matrices. The displacement and rotation at any point is given by [12]

$$\begin{aligned} u(x, y, z) &= u_0(x, y) + z\theta_x, \\ v(x, y, z) &= v_0(x, y) + z\theta_y, \\ w &= w_0 \\ \phi_x &= \theta_x - w_{,x}, \quad \phi_y = \theta_y - w_{,y} \end{aligned} \quad (3)$$

Here, $u_0(x, y), v_0(x, y)$ and w_0 are corresponding mid plane displacements. θ_x and θ_y are rotations along x axis and y axis respectively. ϕ_x and ϕ_y are shear strain in x and y directions. linear strains at any point are expressed as [12].

$$\begin{aligned} \epsilon_x &= u_{,x} = u_{0,x} + z\theta_{x,x} = \epsilon_x^0 + zK_x \\ \epsilon_y &= v_{,y} = v_{0,y} + z\theta_{y,y} = \epsilon_y^0 + zK_y \\ \gamma_{xy} &= u_{,y} + v_{,x} = u_{0,y} + v_{0,x} + z(\theta_{x,y} + \theta_{y,x}) = \epsilon_{xy}^0 + zK_{xy} \\ \gamma_{xz} &= \phi_x; \gamma_{yz} = \phi_y; \epsilon_z = 0 \end{aligned} \quad (4)$$

where, K_x and K_y are the curvatures in x - z and y - z planes, respectively, while K_{xy} is the cross curvature in x - y plane.

The constitutive matrix $[D]$ is given by [12, 13]:

$$[D] = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45} & A_{55} \end{bmatrix}$$

$$\text{Here, } A_{ij}, B_{ij}, D_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} c_{ij} [1, z, z^2] dz \quad (i, j = 1, 2, 6)$$

$$\text{and } A_{ij} = \alpha \int_{-\frac{h}{2}}^{\frac{h}{2}} c_{ij} dz \quad (i, j = 4, 5)$$

Here α is the shear correction factor taken as 5/6 [14].

2.3. Finite Element Formulation

Using nine noded isoparametric element, the linear stiffness matrix is given by

$$[K_e] = \int_{-1}^1 \int_{-1}^1 [B]^T [D] [B] J d\xi d\eta \quad (5)$$

where $[B_i]$ is linear strain-displacement matrix given by [14],

$$[B_i] = \begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 & 0 \\ N_{i,y} & N_{i,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & N_{i,x} & 0 \\ 0 & 0 & 0 & 0 & N_{i,y} \\ 0 & 0 & 0 & N_{i,y} & N_{i,x} \\ 0 & 0 & N_{i,y} & 0 & N_{i,x} \\ 0 & 0 & N_{i,x} & N_{i,y} & 0 \end{bmatrix} \quad (i = 1 \text{ to } 9) \quad (6)$$

The element mass matrix with rotary inertia is given by [13],

$$[M_e] = \iint [N]^T [\rho] [N] J d\xi d\eta \quad (7)$$

in which, inertia matrix

$$[\rho] = \begin{bmatrix} I & & & & \\ 0 & I & & & \\ & 0 & I & & \\ P & 0 & 0 & Q & \\ 0 & P & 0 & 0 & Q \end{bmatrix}, \text{ where,}$$

$$I, P, Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z) (1, Z, Z^2) dz \quad (8)$$

$\rho(z)$ being the density at a distance z . N_i = Shape function at a node i . ξ, η = Local natural co-ordinates of the element.

The governing equation for free vibration analysis is given by

$$([K] - \omega_n^2 [M])\{\delta\} = 0 \quad (9)$$

3. Numerical Study

A FORTRAN program with nine noded lagrangian plate bending element is developed for this study. Study of validation with published literature is presented in this section. After verification of the proposed formulation, the program is used to study the dynamic behaviour of FGM moderately thick rectangular plate structure.

3.1. Validation Study

In this section, the first three non-dimensional natural frequencies of an all side clamped (CCCC) FG square plate (1m X 1m) made up of Al/ZrO₂ is found out for different power law index (N) and side to thickness ratio (a/h) and compared with Talha et al. [8] in Table 2. Another case of FG square plate made up of (Al/Al₂O₃) with different boundary condition (SCSC-two opposite sides are clamped and rest are simply supported) have been studied and compared with Hashemi et al. [9] in Table 3. It is found that the result matches fully with [9] as seen in Table 3 though there is a small discrepancy when compared with [8] in Table 2. The reason is Talha *et al.* [8] have used higher order shear deformation theory (HSDT) whereas Hashemi *et al.* [9] used first order shear deformation theory in their formulation.

Table-2. Variation of the non-dimensional frequency parameter ($\bar{\omega} = \sqrt{12(1 - \nu^2)\rho_c \omega^2 a^2 b^2 / \pi^4 E_c h^2}$) of CCCC square Al/ZrO₂ FG plate

a/h	Mode	N=0		0.5		1		5		10		N=∞	
		PRESENT	[8]	PRESENT	[8]	PRESENT	[8]	PRESENT	[8]	PRESENT	[8]	PRESENT	[8]
10	1	3.45	3.42	3.12	3.09	2.97	2.94	2.76	2.72	2.69	2.65	2.48	2.48
	2	6.59	6.7	5.96	6.08	5.68	5.79	5.25	5.3	5.1	5.15	4.73	4.87
	3	6.59	6.7	5.96	6.08	5.68	5.79	5.25	5.3	5.1	5.15	4.73	4.87
20	1	3.72	3.73	3.34	3.37	3.19	3.21	2.98	2.99	2.90	2.91	2.66	2.72
	2	7.43	7.82	6.69	7.07	6.38	6.73	5.96	6.24	5.79	6.07	5.32	5.69
	3	7.43	7.82	6.69	7.07	6.38	6.73	5.96	6.24	5.79	6.07	5.32	5.69

Table-3. Variation of the non-dimensional frequency parameter ($\beta = \omega a^2 \sqrt{\rho_c / E_c} / h$) for SCSC square Al/Al₂O₃ plate

a/h	MODE	N=0		0.5		1		5		10	
		PRESENT	[9]	PRESENT	[9]	PRESENT	[9]	PRESENT	[9]	PRESENT	[9]
10	1	8.07	8.07	6.88	6.88	6.22	6.22	5.65	5.65	5.29	5.29
	2	14.87	14.90	12.69	12.70	11.46	11.50	10.39	10.40	9.72	9.73
	3	17.93	17.90	15.37	15.40	13.91	13.90	12.61	12.60	11.71	11.70
20	1	8.57	8.57	7.27	7.27	6.56	6.56	5.96	5.96	5.63	5.63
	2	16.07	16.10	13.64	13.60	12.30	12.30	11.18	11.20	10.55	10.60
	3	20.07	20.10	17.06	17.10	15.40	15.40	13.98	14.00	13.17	13.20

Table-4. Non-dimensional frequency parameter($\beta = \omega a^2 \sqrt{\rho_c/E_c}/h$) of FGM(Al/ZrO₂) for all boundary conditions ($a/b=1$)

BC	h/a	MODE	Al/ZrO ₂						Al/Al ₂ O ₃					
			0	0.5	1	5	10	α	0	0.5	1	5	10	α
CCCC	0.05	1	10.59	9.53	9.09	8.50	8.27	7.59	10.59	8.99	8.11	6.95	6.71	5.38
		2	21.16	19.06	18.17	16.97	16.51	15.17	21.16	17.99	16.24	13.88	13.38	10.76
		3	21.16	19.06	18.17	16.97	16.51	15.17	21.16	17.99	16.24	13.88	13.38	10.76
	0.1	1	9.84	8.88	8.47	7.87	7.65	7.06	9.84	8.41	7.60	6.45	6.18	5.01
		2	18.79	16.99	16.18	14.96	14.54	13.47	18.79	16.11	14.59	12.26	11.70	9.55
		3	18.79	16.99	16.18	14.96	14.54	13.47	18.79	16.11	14.59	12.26	11.70	9.55
	0.2	1	8.03	7.29	6.94	6.36	6.17	5.75	8.03	6.95	6.31	5.23	4.94	4.08
		2	14.02	12.75	12.14	11.04	10.71	10.05	14.02	12.20	11.11	9.06	8.52	7.13
		3	14.02	12.75	12.14	11.04	10.71	10.05	14.02	12.20	11.11	9.06	8.52	7.13
CCCF	0.05	1	7.08	6.37	6.07	5.69	5.53	5.07	7.08	6.00	5.41	4.65	4.49	3.60
		2	11.68	10.52	10.03	9.38	9.13	8.37	11.68	9.92	8.95	7.67	7.40	5.94
		3	18.31	16.49	15.72	14.69	14.29	13.13	18.31	15.56	14.04	12.01	11.58	9.31
	0.1	1	6.69	6.03	5.75	5.35	5.21	4.79	6.69	5.70	5.15	4.38	4.21	3.40
		2	10.78	9.73	9.27	8.62	8.38	7.73	10.78	9.21	8.32	7.04	6.76	5.48
		3	16.42	14.84	14.14	13.09	12.72	11.77	16.42	14.07	12.74	10.72	10.24	8.35
	0.2	1	5.64	5.11	4.87	4.48	4.35	4.04	5.64	4.87	4.42	3.67	3.49	2.87
		2	8.71	7.89	7.51	6.89	6.70	6.24	8.71	7.50	6.79	5.60	5.33	4.43
		3	11.90	10.97	10.44	9.24	8.93	8.53	11.90	10.68	9.81	7.72	7.05	6.05
CFCF	0.05	1	6.58	5.92	5.64	5.28	5.14	4.71	6.58	5.58	5.03	4.32	4.18	3.34
		2	7.76	6.99	6.66	6.24	6.07	5.57	7.76	6.59	5.94	5.10	4.93	3.95
		3	12.68	11.41	10.88	10.18	9.90	9.09	12.68	10.76	9.70	8.32	8.03	6.45
	0.1	1	6.24	5.63	5.37	5.00	4.86	4.48	6.24	5.32	4.81	4.09	3.94	3.18
		2	7.27	6.56	6.25	5.82	5.66	5.21	7.27	6.20	5.60	4.76	4.57	3.70
		3	11.67	10.52	10.03	9.32	9.07	8.36	11.67	9.95	8.98	7.60	7.31	5.93
	0.2	1	5.31	4.81	4.58	4.21	4.09	3.81	5.31	4.58	4.16	3.47	3.29	2.70
		2	6.02	5.46	5.19	4.77	4.63	4.31	6.02	5.19	4.70	3.89	3.70	3.06
		3	8.88	8.18	7.79	6.89	6.66	6.37	8.88	7.97	7.31	5.77	5.26	4.52
CFFF	0.05	1	1.05	0.94	0.90	0.84	0.82	0.75	1.05	0.89	0.80	0.69	0.67	0.53
		2	2.52	2.27	2.16	2.03	1.97	1.81	2.52	2.14	1.93	1.66	1.60	1.28
		3	6.34	5.70	5.43	5.09	4.96	4.54	6.34	5.37	4.84	4.16	4.03	3.22
	0.1	1	1.04	0.93	0.89	0.83	0.81	0.74	1.04	0.88	0.79	0.68	0.66	0.53
		2	2.44	2.20	2.09	1.96	1.90	1.75	2.44	2.07	1.87	1.60	1.54	1.24
		3	6.08	5.48	5.22	4.87	4.74	4.36	6.08	5.17	4.66	3.98	3.84	3.09
	0.2	1	1.01	0.91	0.87	0.81	0.79	0.72	1.01	0.86	0.77	0.66	0.64	0.51
		2	2.22	2.01	1.91	1.78	1.73	1.59	2.22	1.90	1.71	1.45	1.39	1.13
		3	3.29	3.03	2.89	2.56	2.47	2.36	3.29	2.96	2.75	2.15	1.95	1.67

3.2. Case study 1: Study of square FG plate ($a/b = 1$)

In this section, the behaviour of a square FG plate (1m X 1m) is studied for variation in thicknesses and power law index N . Different boundary conditions taken in the analysis are as follows:

CCCC- All side clamped

CFCF – Two opposite sides clamped

CCCF – One edge free, Other three edges clamped

CFFF – Clamped only at one edge

Two different materials i.e., Al/ZrO₂ and Al/Al₂O₃ have been chosen in the analysis. The non-dimensional frequency

parameters $\left(\beta = \frac{\omega a^2 \sqrt{\rho_c/E_c}}{h}\right)$ are tabulated in Table 4.

From Table 4, it is seen that with increase in value of N , the non-dimensional natural frequency reduces. The rate of reduction is more for Al/Al₂O₃. With increase in h/a ratio, stiffness increases. Hence natural frequency also increases. non-dimensional frequency decreases, as the thickness term is present in the denominator. CCCC case gives highest stiffness as degree of edge constraints is more. CFFF or cantilever case gives the least frequency. For square plate with symmetrical boundary conditions (CCCC) the second and third mode frequencies are same. This is not seen for other non-symmetric boundary conditions.

3.3. Case study 2: Study of rectangular FG plate ($a/b = 2$)

In this section, the behaviour of a rectangular FG plate (2m X 1m) is studied for variation in thicknesses and power law index N . Different boundary conditions taken in the analysis are as follows:

CCCC- All side clamped

CFCF – Only longer sides clamped

CCCF – One short edge free, Other three edges clamped

FCCC – One long edge free, Other three edges clamped

CFFF – Clamped only at the longer edge

FCFC – Two short edges clamped

Two different materials i.e., Al/ZrO₂ and Al/Al₂O₃ have been chosen in the analysis. The non-dimensional frequency parameters are tabulated in Table 5. First three mode shapes for rectangular plate have been plotted in Fig. 3. Non-dimensional frequency reduces in the following order-CCCC, CCCF, CFCF, FCCC, FCFC, CFFF. It is observed that with increase in edge constraints natural frequency increases. For CCCF and CFCF the difference in frequency

is very less. In CCCF only one short edge is made clamped from free (CFCF) case, thus do not contribute much in the stiffness calculation. From the mode shape it is seen that mainly bending occurs in the structure except for mode 2 in CFFF case where a twisting mode shape is observed.

1. Conclusion

In this paper free vibration behaviour of rectangular FGM plate using First order shear deformation theory has been performed. Different boundary conditions combining free and clamped edges are taken into considerations. Validation study is carried out to verify the accuracy of the present formulation. It is observed that with increase in power law index N , natural frequency decreases exponentially.

The rate of decrease is sharp for Al/Al₂O₃. With increase in thickness by side ratio, stiffness of the structure and hence natural frequency increases. As a result, non-dimensional frequency parameter decreases. Increase in edge constraints also increases stiffness of the structure as expected.

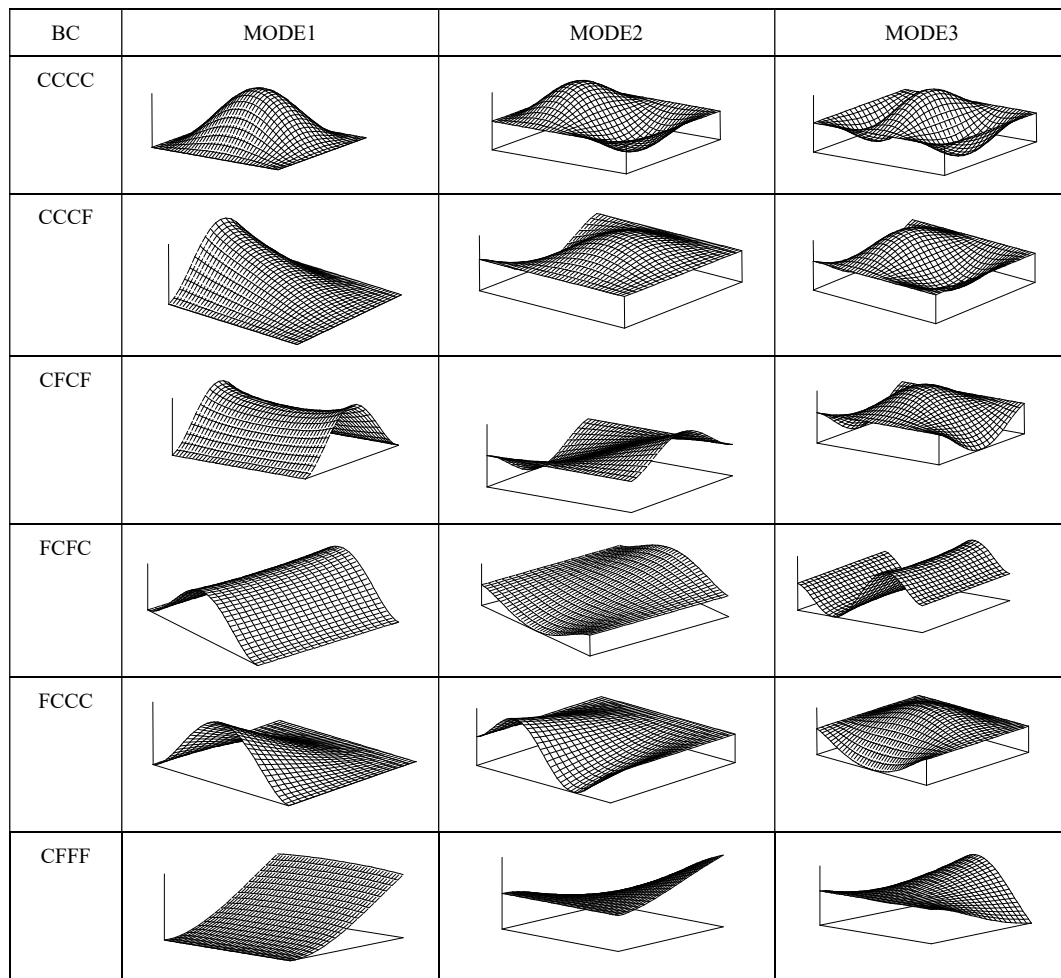


Fig. 3. First three mode shapes for rectangular FG (Al/ZrO₂) plate ($a/b = 2$) for $h/a=0.1$, $N=1$

Table-5. Non-dimensional frequency parameter($\beta = \omega a^2 \sqrt{\rho_c/E_c}/h$) of FGM (Al/ZrO₂) for all boundary conditions ($a/b = 2$)

BC	h/a	MODE	Al/ZrO ₂						Al/Al ₂ O ₃					
			0	0.5	1	5	10	∞	0	0.5	1	5	10	∞
CCCC	0.05	1	27.59	24.88	23.72	22.09	21.48	19.78	27.59	23.52	21.25	18.09	17.38	14.03
		2	35.26	31.81	30.32	28.21	27.44	25.28	35.26	30.08	27.18	23.09	22.18	17.93
		3	48.64	43.90	41.83	38.87	37.80	34.87	48.64	41.54	37.54	31.81	30.51	24.74
	0.1	1	23.29	21.12	20.12	18.49	17.95	16.70	23.29	20.10	18.26	15.19	14.40	11.85
		2	29.30	26.57	25.31	23.24	22.57	21.00	29.30	25.30	22.96	19.07	18.08	14.90
		3	39.31	35.67	33.97	31.12	30.22	28.18	39.31	33.99	30.86	25.51	24.16	19.99
	0.2	1	16.10	14.71	14.00	12.64	12.25	11.54	16.10	14.14	12.94	10.45	9.73	8.19
		2	20.02	18.27	17.39	15.71	15.23	14.35	20.02	17.55	16.03	12.91	12.06	10.18
		3	25.10	23.07	21.91	19.39	18.78	17.99	25.10	22.35	20.50	15.82	14.59	12.77
CCCF	0.05	1	25.47	22.97	21.90	20.40	19.84	18.26	25.47	21.71	19.62	16.70	16.05	12.96
		2	29.00	26.15	24.93	23.21	22.57	20.79	29.00	24.73	22.34	19.00	18.25	14.75
		3	37.11	33.47	31.90	29.68	28.87	26.60	37.11	31.66	28.60	24.28	23.32	18.87
	0.1	1	21.60	19.58	18.65	17.14	16.65	15.48	21.60	18.64	16.93	14.09	13.36	10.98
		2	24.29	22.02	20.98	19.27	18.72	17.41	24.29	20.96	19.02	15.80	14.99	12.35
		3	30.65	27.78	26.45	24.30	23.61	21.97	30.65	26.43	23.95	19.86	18.87	15.59
	0.2	1	14.92	13.63	12.97	11.70	11.34	10.69	14.92	13.11	11.99	9.67	9.00	7.59
		2	16.68	15.20	14.46	13.07	12.69	11.96	16.68	14.57	13.26	10.65	9.99	8.48
		3	20.19	18.53	17.59	15.61	15.11	14.47	20.19	17.87	16.25	12.78	11.80	10.27
CFCF	0.05	1	25.09	22.63	21.57	20.10	19.54	17.99	25.09	21.39	19.32	16.46	15.82	12.76
		2	26.19	23.62	22.51	20.97	20.39	18.77	26.19	22.33	20.17	17.17	16.50	13.32
		3	30.42	27.44	26.15	24.34	23.67	21.81	30.42	25.95	23.44	19.92	19.13	15.47
	0.1	1	21.32	19.33	18.41	16.92	16.43	15.28	21.32	18.40	16.71	13.92	13.19	10.84
		2	22.10	20.03	19.08	17.54	17.03	15.84	22.10	19.07	17.31	14.40	13.65	11.24
		3	25.30	22.93	21.84	20.06	19.49	18.14	25.30	21.82	19.78	16.42	15.59	12.87
	0.2	1	14.74	13.47	12.83	11.57	11.21	10.57	14.74	12.97	11.88	9.58	8.91	7.50
		2	15.19	13.86	13.19	11.91	11.55	10.89	15.19	13.31	12.14	9.75	9.12	7.73
		3	17.30	15.74	14.96	13.56	13.18	12.40	17.30	15.05	13.63	10.95	10.34	8.80
CFFF	0.05	1	4.19	3.77	3.59	3.37	3.28	3.00	4.19	3.55	3.20	2.75	2.66	2.13
		2	6.28	5.66	5.39	5.05	4.91	4.50	6.28	5.33	4.81	4.13	3.99	3.20
		3	11.80	10.62	10.12	9.47	9.22	8.46	11.80	10.02	9.03	7.74	7.47	6.00
	0.1	1	4.08	3.67	3.50	3.27	3.19	2.92	4.08	3.46	3.12	2.68	2.58	2.07
		2	5.95	5.37	5.11	4.77	4.64	4.27	5.95	5.07	4.57	3.90	3.75	3.03
		3	10.90	9.83	9.36	8.71	8.48	7.81	10.90	9.29	8.38	7.10	6.83	5.54
	0.2	1	3.74	3.38	3.22	2.99	2.91	2.68	3.74	3.19	2.88	2.44	2.34	1.90
		2	5.14	4.64	4.42	4.08	3.97	3.68	5.14	4.40	3.98	3.33	3.18	2.61
		3	8.18	7.53	7.17	6.34	6.13	5.86	8.18	7.33	6.78	5.29	4.83	4.16
FCCC	0.05	1	9.10	8.19	7.81	7.31	7.11	6.52	9.10	7.73	6.97	5.98	5.77	4.63
		2	20.12	18.13	17.28	16.14	15.70	14.42	20.12	17.11	15.44	13.20	12.72	10.23
		3	29.12	26.24	25.01	23.33	22.70	20.87	29.12	24.78	22.35	19.05	18.35	14.81
	0.1	1	8.48	7.65	7.29	6.78	6.60	6.08	8.48	7.24	6.54	5.55	5.33	4.31
		2	17.87	16.15	15.39	14.23	13.83	12.81	17.87	15.32	13.86	11.65	11.13	9.09
		3	24.96	22.57	21.48	19.83	19.28	17.89	24.96	21.40	19.32	16.09	15.39	12.69
	0.2	1	7.03	6.37	6.06	5.58	5.42	5.04	7.03	6.06	5.49	4.56	4.34	3.57
		2	13.37	12.16	11.57	10.54	10.23	9.59	13.37	11.62	10.55	8.60	8.12	6.80
		3	17.56	15.74	14.84	13.38	13.11	12.59	17.56	14.87	13.28	10.44	9.88	8.93
FCFC	0.05	1	6.53	5.88	5.60	5.25	5.11	4.68	6.53	5.54	5.00	4.29	4.15	3.32
		2	10.33	9.30	8.87	8.29	8.07	7.41	10.33	8.78	7.92	6.78	6.54	5.25
		3	17.60	15.85	15.11	14.12	13.74	12.62	17.60	14.96	13.50	11.55	11.14	8.95
	0.1	1	6.20	5.59	5.33	4.96	4.83	4.44	6.20	5.28	4.77	4.07	3.91	3.15
		2	9.37	8.46	8.06	7.48	7.27	6.72	9.37	8.00	7.22	6.09	5.85	4.77
		3	15.84	14.31	13.64	12.30	11.89	11.35	15.84	13.57	12.28	10.34	9.41	8.05
	0.2	1	5.28	4.78	4.56	4.19	4.07	3.78	5.28	4.55	4.13	3.44	3.27	2.68
		2	7.29	6.57	6.22	5.68	5.55	5.22	7.29	6.20	5.55	4.45	4.25	3.71
		3	7.92	7.35	7.04	6.24	5.99	5.68	7.92	7.22	6.79	5.45	4.94	4.03

Disclosures

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